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THE USE OF DOTS AS BRACKETS IN CHURCH'S SYSTEM

A. M. TURING

Any logical system, if its use is to be carried beyond a rather elementary stage, needs powerful conventions about abbreviations: in particular one usually wants to modify the bracketing so as to make the formulae more readable, and also possibly shorter. The present note has been written in the belief that Church's formulation of the simple theory of types[1] is particularly suitable as a basis for work on that theory, and that it is therefore worth while introducing special conventions which take into account the needs of this particular system. The conventions which I shall describe are ones which I have used a good deal myself, and have always found adequate. I intend to make use of them in forthcoming papers.[2] They may be regarded as an extension of Curry's conventions.[3]

I shall begin with a general discussion of punctuation by means of groups of dots. This general theory is applicable, with some modifications, to Russell's,[4] Quine's,[5] and Curry's[3] bracketing systems as well as to the present one.

General bracketing theory. We consider a logical system in which every formula is either:

An irreducible formula (or *token* in Curry's terminology).

Of form $\boldsymbol{R}(\boldsymbol{A})$ where $\boldsymbol{R}$ is a monadic operator and $\boldsymbol{A}$ a formula.

Of form $(\boldsymbol{A})\boldsymbol{S}(\boldsymbol{B})$ where $\boldsymbol{S}$ is a dyadic operator and $\boldsymbol{A}$ and $\boldsymbol{B}$ are formulae.

We need not of course enquire further into the nature of the irreducible formulae, monadic operators, and dyadic operators, but to fix our ideas we may think of irreducible formulae as consisting of a single letter with suffixes etc., e.g. x_α, $J^{\alpha,\beta}_{\beta(o\alpha)}$. Typical of monadic operators would be $\sim$, $[\exists x_\alpha]$ and of dyadic operators $\supset$ and $=$. The formulae in this sense will be described in future as *unabbreviated formulae*: the word 'formula' without qualification will be liable to be used of various kinds of series of symbols.

We may also recognise another kind of formulae which we call *abbreviated formulae* and which consist of series of symbols which are irreducible formulae, brackets, monadic and dyadic operators, and a new kind of symbol called a point, which may be thought of as a group of dots. To be an abbreviated formula the series of symbols must satisfy the conditions:

(a) The brackets must be properly paired, i.e., if we go on removing pairs of brackets which face each other and have no other brackets between them there should eventually be no brackets left. The brackets appearing in an abbreviated formula will often be described as 'explicitly shown brackets.'

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[1] Alonzo Church, *A formulation of the simple theory of types*, this JOURNAL, vol. 5 (1940), pp. 56–68.

[2] A. M. Turing, *Some theorems about Church's system*, and *The theory of virtual types*, forthcoming.

[3] H. B. Curry, *On the use of dots as brackets in logical expressions*, this JOURNAL, vol. 2 (1937), pp. 26–28.

[4] Whitehead and Russell, ***Principia mathematica***, vol. 1, pp. 9–11.

[5] W. V. Quine, ***Mathematical logic*** (New York 1940), pp. 37–42.

(b) Of a pair of brackets one must occur adjacent to an operator and one not. The expression 'adjacent to an operator' is used here and elsewhere to mean 'adjacent to a dyadic operator or adjacent to and on the right of a monadic operator.'

(c) If in the formula we replace dyadic operators by 'D', monadic operators by 'M', irreducible formulae by 'x' and points by '**:**', calling the result the 'projected formula,' then the first symbol of a projected abbreviated formula must be '(', 'x', or 'M' and the last, ')' or 'x'. A pair of consecutive symbols in the projected formula must be 'x)', '(x', '))', 'M(', 'D(', ')D', '(M' or '((' or else part of one of the following series of three: 'x**:**D', 'D**:**x', 'M**:**x', ')**:**D', 'D**:**(', 'M**:**(', 'D**:**M', 'M**:**M': in the latter case the whole series of three symbols must be part of the projected formula.

We want one and only one formula to correspond to each abbreviated formula. Such a correspondence is defined below in terms of an ordering of the points. I shall follow Russell's terminology and speak of the earlier of two points in the ordering as being *of higher power* than the other. Curry uses the expression 'senior to' and Quine, whose points are called 'joints,' uses 'looser than.' The power of a point may depend on any formal relationships between the point and the formula it occurs in, and varies from system to system.

The rule for replacing the abbreviated formula by the unabbreviated may be put into two forms, of which the first is the more natural theoretically, and the second, which seems rather arbitrary, is the easier to apply.

First form of rule. The rule operates by reducing the number of points in the formula whose unabbreviated form is to be found.

Suppose first that the formula has explicit brackets, e.g. that it is of form $\boldsymbol{A}(\boldsymbol{B})\boldsymbol{C}$, where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are not required to be formulae in any special sense, but just rows of symbols, and the brackets shown are properly paired. Then the unabbreviated form of $\boldsymbol{A}(\boldsymbol{B})\boldsymbol{C}$ may be obtained from the unabbreviated forms $\boldsymbol{E}$ of $\boldsymbol{A}\boldsymbol{w}\boldsymbol{C}$ and $\boldsymbol{F}$ of $\boldsymbol{B}$ by substituting ($\boldsymbol{F}$) for $\boldsymbol{w}$ in $\boldsymbol{E}$. The symbol $\boldsymbol{w}$ is to be some symbol not occurring in $\boldsymbol{A}$ or $\boldsymbol{C}$. In other words the interior of an explicitly shown bracket is to be worked out as if it were a whole formula, and the part of the formula outside the bracket is to be worked out as if the bracketed part were a single letter.

If the formula has no explicitly shown brackets we find the point of highest power and replace it by a bracket. This bracket is to be right facing if the point is right facing, i.e., if it is on the right of its operator: similarly the bracket is left facing if the point is left facing. Another bracket, oppositely facing, must be put at one end of the formula to balance the first.

Second form of rule. We first define the *enclosing brackets* of a symbol other than an explicitly shown bracket. They are paired explicitly shown brackets, enclosing the symbol in question, but not enclosing any other pair of brackets which enclose the symbol. If the enclosing brackets are always to be defined there must be a pair of brackets enclosing the whole formula. We imagine these supplied.

To find the unabbreviated formula we clearly have to replace each point by a similarly facing bracket, and to put in a balancing bracket somewhere. The interval from the point to the balancing bracket is called the *scope* of the point: in reckoning scopes, points and brackets will be neglected, so that for instance if two similarly facing points are to have their partnering brackets immediately following one another their scopes will be regarded as ending at the same place. The rule for determining the scope is that it is to be as short as possible, subject to the followiug *scope condition*:

The balancing bracket β of a point π is either adjacent to one of the enclosing brackets of π, or else to some point ρ facing oppositely to π and having the same enclosing brackets as π, in which case β must be on the side of ρ which is nearer to π. The point ρ must be of higher power than π or any point between ρ and π facing similarly to π and having the same enclosing brackets as π.

Equivalence theorem. There are three things to be proved about these rules:

(i) When we use the first rule it does not matter in what order the pairs of explicit brackets are taken.

(ii) The result of applying the first rule to an 'abbreviated formula' (satisfying by definition conditions (a), (b), (c) above) is to give us an 'unabbreviated formula' as originally defined.

(iii) The two rules are equivalent.

To prove (i) let $\boldsymbol{A}(\boldsymbol{B})\boldsymbol{C}$ be one of the shortest formulae for which the result of applying the rule is not unique. We are justified in assuming that explicit brackets occur for otherwise the first step in applying the rule is uniquely determined and consists in introducing brackets. Whatever transformation we apply to the formula it remains of the form $\boldsymbol{A}'(\boldsymbol{B}')\boldsymbol{C}'$ where $\boldsymbol{A}'\boldsymbol{w}\boldsymbol{C}'$ is obtained from $\boldsymbol{A}\boldsymbol{w}\boldsymbol{C}$ and $\boldsymbol{B}'$ from $\boldsymbol{B}$ by a (possibly incomplete) application of the rule. In particular this is true of the final result of applying the rule. In this case $\boldsymbol{A}'\boldsymbol{w}\boldsymbol{C}'$ contains no points: it is therefore the final result of applying the rule to $\boldsymbol{A}\boldsymbol{w}\boldsymbol{C}$ and since $\boldsymbol{A}\boldsymbol{w}\boldsymbol{C}$ is shorter than $\boldsymbol{A}(\boldsymbol{B})\boldsymbol{C}$ the formula must be unique. Similarly $\boldsymbol{B}'$ is unique, and therefore $\boldsymbol{A}'(\boldsymbol{B}')\boldsymbol{C}'$ is unique.—The word 'shortest' as used in this argument must be interpreted as 'having the smallest number of symbols, points however being reckoned as two symbols.'

To prove (ii) it is sufficient to show that the application of the transformations described in the rule always leaves us with an abbreviated formula, and that if an abbreviated formula has no points then it is an unabbreviated formula. The transformations always consist of the removal of a point and the introduction of a pair of brackets. The brackets have no other brackets between them, so that the brackets remain properly paired, i.e., (a) remains satisfied. One of the brackets replaces a point, and therefore, by (c) applied to the original formula, is adjacent to an operator. The other bracket is put in either at the end of the formula or adjacent to a similarly facing bracket, facing away from it. It cannot be adjacent to an operator, for if it were there would have been an operator adjacent to the end of the formula, or to a bracket facing towards it, in the original formula, contradicting (c). This shows that (b) remains true. To show that (c) remains true we have only to notice that when we replace points by similarly facing brackets in the admissible combinations the results are made up of admissible combinations, and that admissible combinations always result when a bracket is introduced at the end of the formula or adjacent to a similarly facing bracket.

To prove the second requirement let us see what condition (c) amounts to when there are no points in the formula. The allowable pairs of symbols in the projected formula are 'x)', 'M(', 'D(', ')D', '))', '(x', '(M', '((' and a formula must start with '(', 'M', or 'x' and end with 'x' or ')'. If it starts with 'x' it can only continue with ')' and this bracket can have no partner: i.e., if the projected formula starts with 'x' then 'x' is the whole of it. If it starts with 'M' it continues with '(', and this bracket has a partner, so that the whole is of form $M(\boldsymbol{A})\boldsymbol{B}$, and by (b) of form $M(\boldsymbol{A})$. If the formula starts with '(' this has a partner which by (b) is adjacent to an operator: i.e., the formula is of form $(\boldsymbol{A})D\boldsymbol{B}$ and therefore of form $(\boldsymbol{A})D(\boldsymbol{C})\boldsymbol{E}$. Applying (b) we see it is of form $(\boldsymbol{A})D(\boldsymbol{C})$. Thus we have shown that abbreviated formulae without points are always either irreducible formulae or of one of the forms $\boldsymbol{R}(\boldsymbol{A})$ or $(\boldsymbol{A})\boldsymbol{S}(\boldsymbol{B})$, where $\boldsymbol{R}$ is a monadic and $\boldsymbol{S}$ a dyadic operator. The formulae $\boldsymbol{A}$ and $\boldsymbol{B}$ necessarily satisfy the conditions (a), (b), (c) since the whole formula satisfies them, and the symbols allowed at the ends of a formula by (c) are just the ones which may follow a right facing bracket or precede a left facing bracket: these formulae are therefore themselves 'abbreviated formulae.' An induction over the length of the formula will now prove that every abbreviated formula without points is an unabbreviated formula, as required.

To prove (iii) notice that the second rule agrees with the first as regards the replacement of the points of highest power, for with either rule we may suppose that the enclosing brackets of the point to be replaced are at the ends of the formula. It will therefore be sufficient to prove that the order of replacement of two points may be interchanged when we are using the second rule.

The case when the two points did not originally have the same enclosing brackets is trivial, for then the replacement of the one point does not alter the set of symbols having the same enclosing brackets as the other, and therefore does not alter its scope. We may therefore suppose that the enclosing brackets of both points are at the ends of the formula. We may also suppose that there are no other brackets in the formula, for if any pair of brackets, together with what is between them, is replaced by a single letter, the scope of neither of the points is altered.

The scopes of two points can never be strictly overlapping. Suppose that the scope of one point is limited by brackets α and β of which α is the one further to the left, and the other by γ and δ of which γ is to the left; also that α is to the left of β and that the scopes strictly overlap, so that the brackets form a figure like this:

$$\underset{\alpha}{(}\ldots\underset{\gamma}{(}\ldots\underset{\beta}{)}\ldots\underset{\delta}{)}$$

The points from which these brackets arise can be either at α and γ, or at α and δ, or at β and γ, or at β and δ. The consideration of the last alternative can be omitted as it is the same as the first apart from interchange of left and right.

In the case that the points are at α and γ the brackets α, β must satisfy the scope condition, so that the point at β must be of higher power than those at α and γ or any right facing point between α and β; in particular it is of higher power than those at γ and between γ and β, and therefore by the scope condition the bracket δ partnering γ must have the same position as β, in which case the scopes do not strictly overlap. Next suppose that the points are at α and δ. Then applying the scope condition to the brackets α and β we find that the point at β is stronger than that at γ, and this means that the scope condition cannot be satisfied for a point at δ whose partnering bracket is at γ. Finally suppose that the points are at β and γ. Applying the scope condition to γ and δ we see that either γ or some right facing point between it and β is of higher power than β: but if this is so the scope condition cannot be satisfied for α and β.

This completes the proof that the scopes of two points can never be strictly overlapping, and we now apply it to the interchange of order of removal of brackets under the second rule. Suppose that the scope of the first point is from α to β, the point itself being at α, which we suppose to be to the left of β, and the scope of the second from γ to δ; γ being to the left of δ, but no assumption being made as to whether the point was at γ or δ. We wish to show that the scope of the first point as calculated by the scope condition is unaltered if the other point is replaced by its brackets γ, δ. To fix our ideas we suppose that the scopes α to β and γ to δ are as calculated before either pair of brackets has been put in. The scope of the first point is certainly unaltered by the replacement of the second in the case that the scopes do not overlap at all, for then neither the points within the interval α to β, nor the left facing point (or possibly bracket) at β can be altered by the introduction of γ and δ, and the application of the scope condition gives exactly the same result for the position of β. As the scopes cannot strictly overlap we must suppose that either the interval α to β is wholly contained in the interval γ to δ or wholly contains it. In the first case the data for the application of the scope condition to the bracket β are again not relevantly altered. If the interval γ to δ is wholly contained in the interval α to β we consider separately the possibilities that β might be moved farther to the right or farther to the left by the introduction of γ and δ. To show that β is not moved farther to the right it will be sufficient to show that the interval still satisfies the scope condition. This is certainly the case, for the effect of the introduction of γ and δ, so far from introducing new right facing points is to enclose some in brackets, thereby as it were disqualifying them, and also to remove the point from which γ and δ themselves arose. To show that β is not moved farther to the left we have to show that there can be no left facing points ρ between α and β which satisfy the scope condition. Such a point would certainly have to be between δ and β, for if it were between γ and δ it would not have the same enclosing brackets as α, and if it were between α and γ the position of β would have been at ρ regardless of whether the brackets γ and δ had been put in or not. If ρ between δ and β satisfies the scope condition, then in the original formula there must have been a right facing point σ either at γ, or in the interval γ to δ, which was more powerful than ρ and less powerful than the point at β. However, as the scope of the bracket γ, δ, if it arises from a point at γ, extends only as far as δ, there must have been a point at δ more powerful than σ and therefore than ρ and all right facing points in the interval α to γ. The original position of β would therefore have been the position of δ. If on the other hand the brackets γ and δ arise from a point at δ, then ρ must have been less powerful than some right facing point σ in the interval γ to δ without the alternative of σ being γ itself. We may suppose that σ is the right facing point of highest power in the interval γ to δ. But then as the bracket from δ extends as far as γ, either the point at δ or some left facing point τ in the interval σ to δ must be of greater power than σ and therefore than ρ: τ would then be of higher power than all right facing points in the interval α to γ and also in the interval γ to δ, and therefore would have been the original position of β.

Jutaxposition and omitted points. In most systems there is some operation which is described simply by juxtaposition, without any special operator. In Church's system this is the application of a function to its argument; in Russell's it is conjunction and in algebra it is multiplication. In such systems the abbreviated formulae will be less restricted than the abbreviated formulae in the sense defined here. It is also usual to omit some of the points in the abbreviated formulae, it being understood that a point is to be introduced wherever one is necessary in order to satisfy the conditions (a), (b), (c), above. The power of such points may be settled at the same time as the other power conventions. There is one matter which has been left doubtful about the introduction of these points. When a pair of brackets is adjacent to operators at each end one of the brackets must be 'protected' from its operator by a point, but only one, in order to satisfy (b); which bracket should it be? The following three rules are equivalent:

(1) One may put in a point in both places. In this case (b) is no longer satisfied, and the final result of removing the points, by either of the rules, leaves an otiose pair of brackets which have to be removed before we have an unabbreviated formula.

(2) Both points are put in and then the weaker one removed.

(3) If the conventions below are adopted one may put the point in after the brackets.

With this practical kind of system, where juxtaposition is used and some points are omitted, the abbreviated formulae do not satisfy the conditions (b), (c) above: they satisfy (a), however, and also (c′) below. To distinguish these formulae from the abbreviated formulae proper I will call them *practical formulae*. The conditions (a), (c′) are necessary and sufficient for being a practical formula.

(c′) No pair of consecutive symbols in the projected formula may be one of the following: '( )', '(:', '(*D*', ':)', '::', '*M*)', '*MD*', '*D*)', '*DD*'. No three consecutive symbols may be '*M*:*D*' or '*D*:*D*'. The projected formula may begin only with '(', '*x*', or '*M*' and may end only with ')' or '*x*'.

From a practical formula we can obtain an abbreviated formula by first introducing an operator ★ to take the place of juxtaposition, and afterwards the omitted points. Wherever a point π is not adjacent to an operator we replace it by '$\pi\star\pi$'. We replace ')(' by ')★(', ')*A*' by ')★*A*', '*A*(' by '*A*★(' and

'AB' by '$A{\star}B$' if A and B are irreducible formulae. We then replace the omitted points. We may use small circles to represent them: thus the sequences 'xD', 'Dx', 'Mx', 'MM', 'DM', 'xM', '$)M$' in a projected formula become '$x_\circ D$', '$D_\circ x$', '$M_\circ x$', '$M_\circ M$', '$D_\circ M$', '$x_\circ M$', '$)_\circ M$'. The last two of these must be again modified by the introduction of $\star$, giving '$x_\circ{\star}_\circ M$' and '$)_\circ{\star}_\circ M$' but the process then comes to an end.

Application to Church's system. In Church's system the irreducible formulae are the variables and other single letter formulae, including, if we wish, abbreviations such as $S_{\iota'\iota'}$. The monadic operators are $\sim$, $[x_\alpha]$, $[\exists x_\alpha]$, $[\iota x_\alpha]$, λx_α, and $[\lambda x_\alpha]$, of which the last two may be regarded as the same so far as the unabbreviated formulae are concerned. The dyadic operators are $\supset$, $\mathbf{v}$, $\equiv$, &, $=$, $\neq$, to which we may add $\star$. If we adopt the conventions of the last section it is only necessary to decide on the relative powers of the points in order that the unabbreviated form of a practical formula should be determined. The conventions recommended are as follows:

We divide the operators into two classes:

Class of high power containing $\supset$, $\mathbf{v}$, &, $\equiv$, $\sim$, $[x_\alpha]$, $[\exists x_\alpha]$, $[\iota x_\alpha]$, $[\lambda x_\alpha]$, $=$, $\neq$, and others which may be added from time to time such as $>$, $<$, /.

Class of low power containing λx_α, $\star$, and others which may be added from time to time such as $+$, $-$.

In the class of high power we distinguish some operators as *handicapped*: these are $=$, $\neq$ (and $>$, $<$). A point adjacent to an operator in the high power class is always of higher power than one in the low power class. In the case of two points adjacent to operators of the same class the one with the greater number of dots is of the higher power, with the provisos that if the operator is handicapped the number of dots must be reduced by one, and that a point which is either omitted or represented by $_\circ$ counts as of 'zero dots.' Amongst points of the same class, and having the same (corrected) number of dots the left facing points are of higher power than the right facing. There is no need to decide which shall be the more powerful of two similarly facing points, since this is irrelevant to the scope condition, but for definiteness let us say that the one on the left is the more powerful.

The 'unabbreviated formula' which results from a 'practical formula' by the application of one of our rules is not strictly speaking a formula of Church's system nor even an abbreviation of one which would be recognised by Church. If A is the unabbreviated formula, and $A^{(D)}$ the corresponding formula recognised by Church, then $A^{(D)}$ is A if it is an irreducible formula, and otherwise is defined inductively by the conditions that:

$$((A){\star}(B))^{(D)} \text{ is } (A^{(D)}B^{(D)}),$$

$$((A)\supset(B))^{(D)} \text{ is } [A^{(D)}\supset B^{(D)}],$$

$$((A)\mathbf{v}(B))^{(D)} \text{ is } [A^{(D)}\mathbf{v}B^{(D)}],$$

etc.;

$$(\sim(A))^{(D)} \text{ is } [\sim A^{(D)}],$$
$$([x_\alpha](A))^{(D)} \text{ is } [(x_\alpha)A^{(D)}],$$
$$\text{etc.;}$$
$$([\lambda x_\alpha](A))^{(D)} \text{ is } (\lambda x_\alpha A^{(D)}),$$
$$(\lambda x_\alpha(A))^{(D)} \text{ is } (\lambda x_\alpha A^{(D)}).$$

Discussion of the conventions. These power conventions appear to differ markedly from the Russell conventions because the operator against which a point is placed is made to be of greater effect in determining the power than the number of dots. However in Russell's system the operators in our class of low power do not occur at all, and the difference must be thought of as a rejection of his distinctions between operators for punctuation purposes, together with a special new treatment of the new operators. Our 'handicap of one dot' convention for $=$, $>$, etc. may however be regarded as taking the place of some of Russell's distinctions.

It is easy to remember which are the operators in the class of high power. They are the ones which normally either operate on propositions or form propositions. The ones which are handicapped are those which form propositions but do not normally operate on propositions. The case of $[\lambda x_\alpha]$ is exceptional, but again it is easy to remember its power because the notation has been made analogous to that of the other high power operators. One would not normally use the form $[\lambda x_\alpha]$ unless it is operating on a proposition.

The reason for adopting our high and low power class conventions is that in practice it is extremely seldom that we want the scope of a bracket starting from one of the low power operators to include one of the high power operators. The low power operators are in fact just the ones that we should use in formalising the mathematical formulae in a mathematical book. We should use the high power operators in formalising the English connecting matter. It is hardly necessary to point out that a bracket in one of the formulae never pairs with one in another formula, with English intervening. Our convention has the desired effect of closing automatically all brackets outstanding in the 'mathematical formulae' before going on to the English text. The reasons for adopting the handicap convention are similar. A bracket starting from an equality sign will not usually enclose another high power operator, although a bracket from an operator of low power will not enclose an equality sign.

The convention by which left facing points are made more powerful than right facing is convenient to complete the conventions, and is also in agreement with two of Church's own conventions, viz. that in the absence of other indication association is to the left, and that in the absence of dots an omitted bracket has the minimum possible scope.

The use of square brackets in connection with some of the operators, e.g. $[\exists x_\alpha]$, is necessary in a theoretical treatment, but it is not suggested that such a notation should be generally adopted. With very few exceptions one can tell whether the round brackets are part of an operator or not. One exception is the formula $(p_{oo})(q_o)$.

Examples. (i) As a first example of the effects of our conventions I shall take a very simple formula and remove the dots by the first rule. The formula which I shall take is $ab.c$ and even this will be found quite sufficiently complicated for the purpose. We must first transform the 'practical formula' into an 'abbreviated formula' by introducing the operator $\star$, and the points $_\circ$. This gives us $a_\circ\star_\circ b.\star.c$. We now take the point of highest power, which is the one following the b and replace it by a bracket facing left, i.e., away from its operator, and balance it with a bracket at the left end, giving us $(a_\circ\star_\circ b)\star.c$. We now have to evaluate separately $a_\circ\star_\circ b$ and $\xi\star.c$. The stronger point in $a_\circ\star_\circ b$ is the left one and this formula is therefore equivalent to $(a)\star_\circ b$, i.e., to the result of substituting (a) for η in the unabbreviated form of $\eta\star_\circ b$, i.e., in $\eta\star(b)$. The unabbreviated form of $(a_\circ\star_\circ b)$ is therefore $((a)\star(b))$: also the unabbreviated form of $\xi\star.c$ is $\xi\star(c)$, and therefore the unabbreviated form of $(a_\circ\star_\circ b)\star.c$ is the result of substituting $((a)\star(b))$ for ξ in $\xi\star(c)$, i.e., is $((a)\star(b))\star(c)$. Transforming this back to a formula of Church's system, properly speaking, we get $((ab)c)$.

In the remaining examples we will always use the second rule. No type suffixes will be shown.

(ii) We will first deal with formulae without operators, or at least without operators of high power. As one example,

$$(a((cd)(e(fg))))$$

can be abbreviated to

$$a\,{:}.\,cd\,{:}\,e.fg.$$

As another,

$$a.cd.efg$$

is an abbreviation of

$$((a(cd))((ef)g)).$$

The association to the left rule has been used here: in other words we have had to apply the rule that a dot is more powerful in its left facing than its right facing aspect. The structure of a formula is often more easily taken in if we slightly increase the number of dots and do not rely on this rule, e.g. the same formula may be written

$$a.cd\,{:}\,efg,$$

or again as

$$a.cd\,{:}\,ef.g.$$

Similarly it is often not advisable to replace all of the brackets in a formula by dots. As a group of dots never replaces more than four brackets it can hardly ever be worth while having as many as six dots, say, in a group. A few dots can however be made to go a long way by mixing them judiciously with explicitly shown brackets, e.g.

$$bc.d::.e::f:.g:h.ij$$

is the best form of a certain formula when expressed without any explicit brackets, but

$$bc.d:e.f(g:h.ij)$$

is a much better form of it.

As an example of a formula involving λ,

$$h:.\lambda f\lambda x.fx:g$$

is an abbreviation of

$$(h((\lambda f(\lambda x(fx)))g)).$$

(iii) As an example of a more general type of formula,

$$[m] \,.\, Nm \supset [p] \,.\, Np \supset m \neq S:pS.m$$

is an abbreviation of

$$[m]((Nm) \supset ([p]((Np) \supset (m \neq S((pS)m)))))).$$

If we did not have the 'handicap of one dot' convention we should have to put in a dot after '$Np \supset$'. In this case the effect is slight, but sometimes it can be considerable, e.g. without the convention

$$[x] \,.\, x=y \;\&\; y=z \supset x=z$$

would have to become

$$[x] : x=y \,.\, \& \,.\, y=z \,.\, \supset \,.\, x=z.$$

(iv) The expressions

$$p \supset . \, q \supset : r \vee s \,.\, \& \, t : \& \, u$$

and

$$p \supset (q \supset ((r \vee s) \;\&\; t)) \;\&\; u$$

and

$$p \supset (q \supset . \, r \vee s \;\&\; t) \;\&\; u$$

are all abbreviations of the same formula. Notice that in the first of these expressions the bracket starting after '$p \supset$' does not close when we reach the stronger point on the left of '& t', because the former is reinforced by the even stronger point after '$q \supset$'. The most legible form of this formula, if it is standing by itself, is probably

$$p \supset : q \supset : r \vee s \,.\, \& \, t : \& \, u.$$

(v) A formula similar to the last example in one respect is

$$p \supset q \;\&\; r,$$

which with our conventions is an abbreviation of

$$(p \supset q) \;\&\; r,$$

but with Russell's or Church's conventions would be an abbreviation of

$$p \supset (q \;\&\; r)$$

on account of the subdivision of our 'class of high power' into smaller classes of different powers.

(vi) Normally we shall not want to put dots against equality signs, or other operators which form propositions but do not operate on propositions. A typical exception is

$$[\iota x_\alpha] \,.\, g_{o\alpha} x_\alpha \supset f_{o\alpha} x_\alpha \;:\; = y_\alpha.$$

Another type of freak formula, difficult to abbreviate, occurs when we have functions which take propositions as arguments, e.g.

$$h_{\alpha o}(p_o \supset q_o).$$

The only way of avoiding explicit brackets in such a case is to express the implication, not with the implication operator but with the implication function C_{ooo}, thus

$$h_{\alpha o}.C_{ooo} p_o q_o.$$

KING'S COLLEGE, CAMBRIDGE